Stat Phys IV
Lecture 2

Reference "The Mathematics of Brownian Motion and Johnson noise"
Gillespie Am. J. Phys. 1996

(Continous Markov Process Theory)

Mathematical Foundations of Langevin Eqs:

ordinary calculus $\qquad \frac{dX(t)}{dt} = A(X(t),t) \qquad$ A: smooth fct.

can be expressed as: (where X(t) is called PROCESS)

$$\boxed{X(t+dt) = X(t) + A(X(t),t)\,dt \quad (1)}$$ "UPDATE FORMULA"

Properties of process:

(1) Deterministic since $X(t+dt)$ is given UNEQUIVOCALLY

(2) CONTINUOUS since $X(t+dt) = X(t)$ for $dt \to 0$

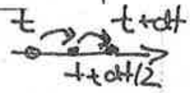
(3) MEMORYLESS since $X(t+dt)$ depends on $X(t), t$.

$\Rightarrow$ CONTINOUS, MEMORYLESS, DETERMINISTIC PROCESS.

Generalization to a CONTINOUS, MEMORYLESS, STOCHASTIC PROCESS $\Rightarrow$ CONTINOUS MARKOV PROCESS.

$$\boxed{\text{Näive approach:} \quad X(t+dt) = X(t) + \sqrt{dt}\, A(X(t),t) \quad (2)}$$

This update formula is not consistent since,



(1) $\quad X(t+dt/2) = X(t) + A(X(t),t)\cdot\sqrt{dt/2}$

(2) $\quad X(t+dt) = X(t+dt/2) + A(X(t+dt/2))\cdot\sqrt{dt/2}$

inserting yields:

$$X(t+dt) = X(t) + A(X(t))\sqrt{dt/2} + A(X(t+dt/2))\cdot\sqrt{dt/2}$$

in the limit of $dt \to 0$

$$X(t+dt) = X(t) + 2\sqrt{\frac{dt}{2}}\,A(X(t),t)$$ $\rightarrow$ different from Equation (2).

The self consistency condition limits all continous, memoryless stochastic processes to evolve under generalizations of (1) "LANGEVIN EQUATIONS".

Note. the condition of a memoryless process is not as restrictive as one may think: example:

e.g reduce $m\ddot{x} = -\frac{\partial V}{dx}$ to 1st order diff eqs. $\begin{cases} m\dot{v} = \frac{\partial V}{\partial x} \\ \dot{x} = v \end{cases}$ BIVARIATE CONT. MEMOR. PROCESS

Review (brief) Random Variabels

$P(Y)$ Probability density fct. of variabel Y ($P(Y)dy$ is prob. to find $Y \dots Y+dy$.)

$$\langle f(Y)\rangle = \lim_{N\to\infty} \sum_{i=1}^{N} f(Y^i) = \int_{-\infty}^{\infty} p(y) f(y)\, dy$$

VARIANCE: $\mathrm{VAR}\{Y\} = \langle (Y-\langle Y\rangle)^2\rangle = \langle Y^2\rangle - \langle Y\rangle^2 \equiv (\langle \Delta Y^2\rangle)$

STANDARD DEVIATION $\sigma \equiv \sqrt{\mathrm{VAR}\{Y\}}$

COVARIANCE
$$\mathrm{COV}\{Y_1, Y_2\} = \langle (Y_1-\bar{Y}_1)(Y_2-\bar{Y}_2)\rangle$$
$$= \langle Y_1 Y_2\rangle - \langle Y_1\rangle\langle Y_2\rangle$$

for statistically independent variabels: $\mathrm{COV}\{Y_1, Y_2\} = 0$

GAUSSIAN RANDOM VARIABEL

$$\boxed{p(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(y-m)}{2\sigma^2}\right)} \qquad Y = N(m, \sigma^2)$$

mean variance

unit random variabel: $N(0,1) \equiv N$

Properties:

- $\alpha + \beta N(m, \sigma^2) = N(\alpha + \beta m\,;\ \beta^2\sigma^2)$

  thus for unit random variabel:
  $$\alpha + \beta N(0,1) = N(\alpha, \beta^2)$$

- $N(m_1, \sigma_1^2) + N(m_2, \sigma_2^2) = N(m_1 + m_2,\ \sigma_1^2 + \sigma_2^2)$

- CENTRAL LIMIT THEOREM

$$\tilde{X} = \sum_{i=1}^{k} X_1^{(1)} + X_2^{(i)} + \dots X_k^{(1)}$$

X_k: statistically independent but not normally distributed

as $k\to\infty$ leads to a NORMAL i.e GAUSSIAN DISTRIBUTION.

DEFINITION: A STOCHASTIC PROCESS $X(t)$ is a random variabel whose propabilty density depends on time t, $X(t_1)$ and $X_2(t_2)$ are two different random variabels.

$X(t)$ is a CONTINOUS, MEMORYLESS, STOCHASTIC PROCESS (i.e A CONTINOUS MARKOV PROCESS) if:

(i) MEMORYLESS i.e $E(dt, x, t) = X(t+dt) - X(t)$
E: increment depends only on t, dt, $X(t)$

(ii) $E(x, t, dt)$ depends "smoothly" on t, x, dt

(iii) CONTINOUS i.e $E(x, t, dt) \to 0$ for $dt \to 0$.

Conditions (i)-(iii) imply that a MEMORYLESS, STOCHASTIC, CONTINOUS MARKOV PROCESS must have the form.

$$\boxed{\Xi(t,x,dt) = \underbrace{A(x(t),t)\,dt}_{\text{Drift-term}} + \underbrace{D^{1/2}}_{\text{Diffusion}}(x(t),t)\,\underbrace{N(t)}_{N(0,1)}\sqrt{dt}} \quad (3)$$

$\langle N^2(t)\rangle = 1$
$\langle N(t)\rangle = 0$

Proof requires as before to consider partial intervals ie $dt = \left(\sum_{i=1}^{N} dt^i\right)$ N: intervals of dt increment.

rewritten as UPDATE FORMULA:

$$\boxed{x(t+dt) = x(t) + A(x(t),t)\,dt + D^{1/2}(x(t),dt)\,N(t)\sqrt{dt}} \quad (4)$$

$D=0$: continous deterministic Markov process (mem-less)
$D\neq 0$: continous stochastic Markov process (mem-less)

X is CONTINUOUS but NOT differentiable since:

$$\frac{x(t+dt)-x(t)}{dt} = A(x(t),t) + D^{1/2}(x(t),t)\,\frac{N(t)}{\sqrt{dt}}$$

diverges as $dt \to 0$.

$$\frac{N(t)}{\sqrt{dt}} = \sqrt{dt}^{\,-1} N(0,1) = N\left(0, \frac{1}{dt}\right)$$

We define GAUSSIAN WHITE NOISE:

$$\boxed{\Gamma(t) \equiv \lim_{dt\to 0} N\left(0, \frac{1}{dt}\right)} \quad (5)$$

rewriting Equation (3) / (4)

$$\boxed{\frac{dX(t)}{dt} = A(x(t),t) + \Gamma(t)\,D^{1/2}(x,t)}$$

White Noise form of the Langevin Eq.

Properties:

$$\boxed{\begin{aligned} &\langle \Gamma(t)\rangle = 0 \\ &\langle \Gamma(t)\Gamma(t')\rangle = \delta(t-t') \\ &\langle X(t')N(t)\rangle = 0 \quad \text{for all } t' \leq t \end{aligned}}$$

- Side remark (comes later in class): $P(x,t)$ ie the probabily density function of the random variabel satisfies the FORWARD FOKKER PLANCK EQS:

$$\partial_t P(x,t) = -\partial_x \left[A(x(t),t)\,P(x,t)\right] + \frac{1}{2}\partial_x^2 \left[D(x,t)\,P(x,t)\right]$$

- Note also that $X(t)$ does not have a proper derivative. BUT

$\frac{dY(t)}{dt} = x(t)$ has since $Y(t+dt) = Y(t) + x(t)dt$

The Ornstein Uhlenbeck process

$A(x,t) = -\frac{1}{T}x$ $\qquad D(x,t) = c$

Important for Brownian motion Johnson Noise Note: $\frac{dV}{dt} = \frac{1}{m\gamma}V + \frac{F(t)}{m}$

Update formula: $\boxed{X(t+dt) = X(t) - \frac{1}{2}X(t)dt + \sqrt{c}\,N(t)\sqrt{dt}}$ standard langevin equation

or $\boxed{\frac{dX}{dt} = -\frac{1}{2}X(t) + \sqrt{c}\cdot N(t)}$

initial condition $X(t_0) = X_0$

(1) Take average $\langle X(t+dt)\rangle = \langle X(t)\rangle - \frac{1}{T}\langle X(t)\rangle dt \quad \wedge \quad \langle N(t)\rangle = 0$

$\Leftrightarrow \frac{d}{dt}\langle X(t)\rangle = -\frac{1}{T}X(t)$

$$\boxed{\langle X(t)\rangle = e^{-(t-t_0)/T}\cdot X_0} \quad \text{MEAN}$$

Next we square ("ITO CALCULUS")

$$X^2(t+dt) = X^2(t) - \frac{2}{T}X(t)^2dt + 2\sqrt{c}\,N(t)X(t)\sqrt{dt} + cN^2(t)$$

average: $\langle X^2(t+dt)\rangle = \langle X^2(t)\rangle - \frac{2}{T}\langle X^2(t)\rangle dt + c\,dt$

$\Rightarrow \frac{d}{dt}\langle X^2(t)\rangle = -\frac{2}{T}\langle X^2 t\rangle + c$

$$\boxed{var\{X(t)\} = \langle \Delta X(t)^2\rangle = \frac{cT}{2}\left(1 - e^{-2(t-t_0)/T}\right)} \quad \text{VARIANCE}$$

Since $X(t)$ is normal we conclude that for $X(t_0) = X_0$

$$\boxed{X(t) = N\left(X_0 e^{-(t-t_0)/T}\,;\ \frac{cT}{2}\left(1-e^{-2(t-t_0)/T}\right)\right) \quad \text{i.e. for } t \gg T \quad N\left(0, \frac{cT}{2}\right) \quad \text{long time limit}}$$

* STATIONARY STATISTICAL PREDICTION

Analogous $cov\{X_1(t_1)X_2(t)\} = \frac{cT}{2}e^{-(t_2-t_1)/T}\left(1-e^{-2(t_1-t_0)/T}\right)$

We may also need sometimes the integral (remember Langevin Eqs is for the velocity not position)

$$\langle Y(t)\rangle = \int_{t_0}^{t} dt'\langle X(t')\rangle \quad ; \quad \langle Y^2(t)\rangle \overset{*}{=} 2\int_{t_0}^{t}\langle Y(t')X(t')\rangle dt'$$

Proof Hw

without proof

$\langle Y(t)\rangle = X_0T\left(1-e^{-(t-t_0)/T}\right) \to X_0T$

not a Stable Process $\quad \langle\langle Y^2\rangle - \langle Y\rangle^2\rangle = VAR\{Y(t)\} = c\cdot T^2\left[\Delta t - 2\left(1-e^{-\Delta t/T}\right) + \frac{1}{2}\left(1-e^{-2\Delta t/T}\right)\right]$

$\to cT^2\Delta t$ NOT STATIONARY

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Wiener Process $A(x,t)=0 \quad D(x,t)=c$

Driftless Wiener process. $x(t+dt) = x(t) + \sqrt{c}\, N(t) \sqrt{dt}$

$\Rightarrow x(t) = N(x_0, c(t-t_0))$ solution

ie the O-U process in the limit of $\tau \to \infty$ becomes a driftless Wiener process. Example: phase diffusion in a LASER

SPECTRAL DENSITY FUNCTION

One advantage of the Markovian description of Noise is the (over thermodynamical ones) is the insights into the freq. spectrum of fluctuations.

$$\boxed{G(\tau) = \langle X(t)X(t+\tau)\rangle = C_x(\tau)}$$ auto-covariance (or autocorrelation)

Then:

$$S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, C_x(\tau)\, d\tau$$ spectrum eg $\left[\frac{m^2}{Hz}\right]$ with $C_x(\tau) \in \mathbb{R}$

$$S_{xx}(\omega) = \frac{1}{\pi}\int_0^{\infty} C_x(\tau) \cos(\underbrace{2\pi\nu}_{\omega}\,\tau)\, d\tau$$ spectral density

ν: frequency
ω: angular frequency.

conversely:
$G(\tau) = G(-\tau)$

$$C_x(\tau) = G(\tau) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\, G(\tau)\, d\tau$$
$$S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, G(\tau)\, d\tau$$

WIENER KHINCHIN THEOREM $\qquad \langle X^2(t)\rangle = \sigma^2$ INTENSITY

it follows $$\boxed{\langle X^2(t)\rangle = \int_0^{\infty} S_{xx}(\nu)\, d\nu}$$ integral over the spectrum gives the VARIANCE

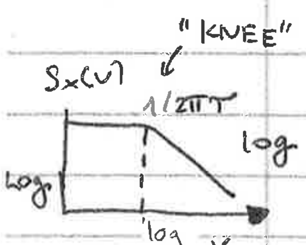
$S_{xx}(\nu)\, d\nu$ is portion of intensity with frequencies $\nu \ldots \nu + d\nu$.

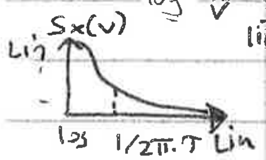
Example

O-U process:

$$C_x(t') = \frac{c\tau}{2} \cdot e^{-t'/\tau}$$ exponentially decaying

$$S_{xx}(\nu) = \frac{2c\tau^2}{1 + (2\pi\nu\tau)^2}$$ SPECTRAL DENSITY spectrum of fluctuations





lin Second Example: Random Force / Function. Gaussian white NOISE

$$C_\Gamma(t') = \langle \Gamma(t)\Gamma(t')\rangle = \delta(t')\, \alpha^2$$
$$S_{\Gamma\Gamma}(\nu) = \alpha^2 \int \delta(t')\, e^{-i\omega t'}\, dt' = \alpha^2$$ $\Rightarrow$ FLAT ie WHITE NOISE

Note spectral density of a wiener process

<u>problem</u>: $\langle x^2(t)\rangle_{wiener} = \langle N_{\substack{0,\\x_0}}(c(t-t_0))^2\rangle$

$= c(t-t_0)$ infinite intensity!

But: $S_{xx}(\nu) = \dfrac{2c\tau^2}{1+(2\pi\tau\nu)^2}$ (for OU process)

in the limit $\tau\to\infty$ (driftless Wiener Process):

$$S_{xx}(\nu) \to \frac{2c}{2\pi\nu^2} \propto \frac{1}{\nu^2} \quad (\text{"}1/f^2\text{" NOISE})$$

only valid for high frequencies since $(t-t_0) \gg \tau$
problem for $\tau\to\infty$

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Finally back to Langevin: TIME EVOLUTION EQU

$\frac{\gamma}{m} = \frac{1}{\tau}$

$F(t) = m\sqrt{c}\,\Gamma(t)$ $$\boxed{\frac{dV}{dt} = -\frac{1}{\tau}V(t) + \sqrt{c}\,\Gamma(t)}$$

- Thus, ~~this is a~~ O-U (Ornstein-Uhlenbeck) process with relaxation time τ and diffusion c. ⇒ MEMORYLESS + ZERO MEAN → ~~SE~~ + CONTINUOUS ⇒ need to be prop to Gaussian white noise

Fixing c and τ:

we have now:

$$\begin{cases} \langle V(t\to\infty)\rangle = N(0, k_BT/m) \\ \langle X^2(t\to\infty)\rangle = 2D(t-t_0) \end{cases}$$

Follow from maxwell Boltzmann distribution $\sigma^2 = \Delta V^2 = k_BT/m$

diffussion behavior from observations

$$\left.\begin{matrix} \langle V(t\to\infty)\rangle = N(0, c\tau/2) \\ \langle x^2(t\to\infty)\rangle = c\tau^2(t-t_0) \end{matrix}\right\}$$ from OU maths

$\boxed{D = \frac{k_BT}{m\gamma}}$

Thus $c\tau/2 = k_BT/m$
$c\tau^2 = 2D$

$\boxed{\tau = \frac{Dm}{k_BT}}$ or $\gamma = \frac{D^{-1}m}{k_BT}$

OR $c = \frac{2}{D}\left(\frac{kT}{m}\right)^2$

Solution $\boxed{V(t) = N\left(v_0 e^{-(t-t_0)/\tau}; \frac{k_BT}{m}\left(1 - e^{-2(t-t_0)/\tau}\right)\right)}$

Thus $\boxed{\langle F(t)F(t')\rangle = \delta(t-t')\, 2k_BT\gamma}$.

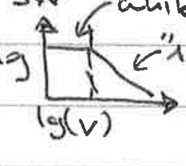
Hence importantly we obtain FLUCTUATION DISSIPATION THEOREM

$$\boxed{\gamma = \frac{1}{2k_BT}\int_{-\infty}^{\infty}\langle F(t)F(t+t')\rangle dt'}$$ NOISE → DISSIPATION

Also $\langle V(t)V(t+t')\rangle = c\tau/2\, e^{-t'/\tau}$

Particles Fluctuation spectrum. $\langle V.F\rangle = P = 3\gamma\frac{k_BT}{m}$ velocity spectrum

$$P(\nu) = \frac{12\,k_BT}{1+\left(\frac{2\pi\nu m}{\gamma}\right)^2}\ ; \quad S_{VV}(\nu) = \frac{k_BT/m\cdot 12}{1+(2\pi m\nu\gamma)^2}$$



(2) Johnson Noise 1928 by Johnson electrical Noise + Nyquist → key parameter R not D.

$$\boxed{-R.I(t) + V(t) - L\frac{dI(t)}{dt} = 0}$$

$\langle V(t)\rangle = 0$
Random voltage
$L/R = \tau$

∗ Fluct. voltage $\boxed{V = L\sqrt{c}\,\Gamma(t)}$

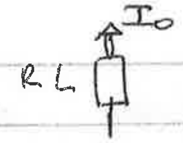
$$-L\langle\frac{dI}{dt}\rangle = R\langle I(t)\rangle$$

$$\Leftrightarrow \frac{dI}{dt} = -\frac{1}{\tau}I + \sqrt{c}.\Gamma(t) \quad \xrightarrow{\times L} ;\text{or: } \frac{dV}{dt} = -\frac{L}{\tau}V + \sqrt{c}V\,\Gamma(t)$$

Fixing c by Boltzmann, $\boxed{k_BT/2 = \langle\frac{1}{2}LI^2\rangle}$

Thus: $c = \frac{2k_BTR}{L^2}$ i.e $\boxed{V(t) = 2k_BTR\,\Gamma(t)}$

Likewise $$\boxed{R = \frac{1}{2k_BT}\int_{-\infty}^{\infty} V(t)V(t+t')dt'}$$ $S_I = \frac{4k_BT}{R}\frac{1}{1+\left(\frac{2\pi L\nu}{R}\right)^2}$ spectrum



$I(t_0) = i_0$: initial condition

$$I(t) = N\left(i_0 e^{-\frac{R}{L}(t-t_0)} ; \frac{kT}{L}\left(1 - e^{-\frac{2R}{L}(t-t_0)}\right)\right)$$